Reinvigorating pen-and-paper proofs in VDM: the pointfree approach

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Formal methods

Adopting a **formal** notation standard such as VDM-SL isn’t enough:

- abstract models involve **conditions** which lead to
- **proof obligations** that need to be discharged

As in other branches of engineering

\[ e = m + c \]

that is,

*engineering = model first, then calculate* ...

Calculate? Verify?

We know how to **calculate** since the school desk...
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\[ \textit{engineering} = \textit{model first, then calculate} \ldots \]

**Calculate? Verify?**

We know how to **calculate** since the school desk...
Tradition on “al-djabr” equational reasoning

Examples of “al-djabr” rules: in arithmetics

\[ x - z \leq y \equiv x \leq y + z \]

In logics:

\[ (x \land \neg z) \Rightarrow y \equiv x \Rightarrow (y \lor z) \]

“Al-djabr” rules are known since the 9c. (They are nowadays known as Galois connections.)

Question

Can VDM proof obligations be calculated along the same tradition?
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By the way

Nunes’ *Libro de Algebra en Arithmetica y Geometria* (1567)

(...*) ho inuëtor desta arte foy hum Mathematico Mouro, cujo nome era Gebre, & ha em algumas Liuarias hum pequeno tractado Arauigo, que contem os capitulos de ã usamos (fol. a ij r)

Reference to *On the calculus of al-gabr and al-muqâbala* \(^1\) by Abû Abd Allâh Muhamad B. Mûsâ Al-Huwârizmî, a famous 9c Persian mathematician________________

\(^1\)Original title: *Kitâb al-muhtasar fi hisab al-gabr wa-almuqâbala*. 
Examples of proof obligations

The following are standard in VDM:

- **Satisfiability**: a pre/post pair is *satisfiable* iff

  \[ \forall a \cdot \text{pre}(a) \Rightarrow \exists b \cdot \text{post}(a, b) \tag{1} \]

- **Invariants**: in case the pre/post pair specifies an operation over a state with invariant \text{inv},

  \[ \forall a \cdot \text{pre}(a) \Rightarrow \exists b \cdot \text{inv}(b) \land \text{post}(a, b) \tag{2} \]

Moreover, invariants are to be maintained:

\[ \forall b, a \cdot \text{pre}(a) \land \text{post}(a, b) \land \text{inv}(a) \Rightarrow \text{inv}(b) \tag{3} \]
Examples of proof obligations

The following are standard in VDM:

- **Satisfiability:** a pre/post pair is *satisfiable* iff

  \[ \forall a \cdot pre(a) \Rightarrow \exists b \cdot post(a, b) \]  

  (1)

- **Invariants:** in case the pre/post pair specifies an operation over a state with invariant \textit{inv},

  \[ \forall a \cdot pre(a) \Rightarrow \exists b \cdot inv(b) \land post(a, b) \]  

  (2)

Moreover, invariants are to be maintained:

\[ \forall b, a \cdot pre(a) \land post(a, b) \land inv(a) \Rightarrow inv(b) \]  

(3)
Impact of (universal) quantification

Quantifiers:

- $\exists$ — easy to discharge (eg. by counter-examples)
- $\forall$ — hard to calculate with (in general), leading to (complex) inductive proofs.

What can we do about this?

- Mechanical proof support is one way
- Investigation of alternative calculation methods is another

An analogy:

$$\langle \forall x : 0 < x < 10 : x^2 \geq x \rangle$$

$$\langle \int x : 0 < x < 10 : x^2 - x \rangle$$

How has traditional engineering mathematics tackled the complexity brought about by $\int$’s and $\frac{\partial}{\partial x}$’s?
Impact of (universal) quantification

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\]
\[
\langle \int \ x : 0 < x < 10 : x^2 - x \rangle
\]

How has traditional **engineering mathematics** tackled the complexity brought about by \( \int \)’s and \( \partial / \partial x \)’s?
The Laplace transform

\[(\mathcal{L} f)(s) = \int_0^\infty e^{-st} f(t)\,dt\]

<table>
<thead>
<tr>
<th>(f(t))</th>
<th>(\mathcal{L}(f))</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>(\frac{1}{s})</td>
</tr>
<tr>
<td>(t)</td>
<td>(\frac{1}{s^2})</td>
</tr>
<tr>
<td>(t^n)</td>
<td>(\frac{n!}{s^{n+1}})</td>
</tr>
<tr>
<td>(e^{at})</td>
<td>(\frac{1}{s-a})</td>
</tr>
<tr>
<td>etc</td>
<td></td>
</tr>
</tbody>
</table>

Pierre Laplace (1749-1827)
**How it works**

Given problem:
\[ y'' + 4y' + 3y = 0 \]
\[ y(0) = 3 \]
\[ y'(0) = 1 \]

Subsidiary equation:
\[ s^2 + 4sY + 3Y = 3s + 13 \]

Solution of given problem:
\[ y(t) = -2e^{-3t} + 5e^{-t} \]

Solution of subs. equation:
\[ Y = \frac{-2}{s+3} + \frac{5}{s+1} \]
An “s-space analog” for logical quantification

The pointfree ($\mathcal{PF}$) transform

<table>
<thead>
<tr>
<th>$\phi$</th>
<th>$\mathcal{PF} \phi$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$\langle \exists \ a :: b \ R \ a \land a \ S \ c \rangle$</td>
<td>$b(R \cdot S)c$</td>
</tr>
<tr>
<td>$\langle \forall \ a, b :: b \ R \ a \Rightarrow b \ S \ a \rangle$</td>
<td>$R \subseteq S$</td>
</tr>
<tr>
<td>$\langle \forall \ a :: a \ R \ a \rangle$</td>
<td>$id \subseteq R$</td>
</tr>
<tr>
<td>$\langle \forall \ x :: x \ R \ b \Rightarrow x \ S \ a \rangle$</td>
<td>$b(R \setminus S)a$</td>
</tr>
<tr>
<td>$\langle \forall \ c :: b \ R \ c \Rightarrow a \ S \ c \rangle$</td>
<td>$a(S / R)b$</td>
</tr>
<tr>
<td>$b \ R \ a \land c \ S \ a$</td>
<td>$(b, c)\langle R, S \rangle a$</td>
</tr>
<tr>
<td>$b \ R \ a \land d \ S \ c$</td>
<td>$(b, d)(R \times S)(a, c)$</td>
</tr>
<tr>
<td>$b \ R \ a \land b \ S \ a$</td>
<td>$b(R \cap S)a$</td>
</tr>
<tr>
<td>$b \ R \ a \lor b \ S \ a$</td>
<td>$b(R \cup S)a$</td>
</tr>
<tr>
<td>$(f \ b) \ R \ (g \ a)$</td>
<td>$b(f^\circ \cdot R \cdot g)a$</td>
</tr>
<tr>
<td>$\text{TRUE}$</td>
<td>$b \top a$</td>
</tr>
<tr>
<td>$\text{FALSE}$</td>
<td>$b \bot a$</td>
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A transform for logic and set-theory

An old idea

\[ \mathcal{PF}(\text{sets, predicates}) = \text{binary relations} \]

Calculus of binary relations

- 1860 - introduced by De Morgan, embryonic
- 1941 - Tarski's school, cf. *A Formalization of Set Theory without Variables*
- 1980's - coreflexive models of sets (Freyd and Scedrov, Eindhoven school)

Unifying approach

*Everything* is a (binary) relation
A transform for logic and set-theory

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Unifying approach

Everything is a (binary) relation
Binary Relations

Arrow notation
Arrow \( A \xrightarrow{R} B \) denotes a binary relation to \( B \) (target) from \( A \) (source).

Identity of composition
\( id \) such that \( R \cdot id = id \cdot R = R \)

Converse
Converse of \( R \) — \( R^\circ \) such that \( a(R^\circ)b \) iff \( b \ R \ a \).

Ordering
“\( R \subseteq S \) — the “\( R \) is at most \( S \)” — the obvious \( R \subseteq S \) ordering.
Binary relation taxonomy

Recall

where a relation $A \xrightarrow{R} A$ is

- reflexive: $\text{iff } id_A \subseteq R$
- coreflexive: $\text{iff } R \subseteq id_A$
- transitive: $\text{iff } R \cdot R \subseteq R$
- anti-symmetric: $\text{iff } R \cap R^\circ \subseteq id_A$
- symmetric: $\text{iff } R \subseteq R^\circ (\equiv R = R^\circ)$
- connected: $\text{iff } R \cup R^\circ = \top$
Binary relation taxonomy

Recall

where a relation \( A \xrightarrow{R} A \) is

- reflexive: \( \iff \text{id}_A \subseteq R \)
- coreflexive: \( \iff R \subseteq \text{id}_A \)
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Binary relation taxonomy

The whole picture:

where

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<th>Coreflexive</th>
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<tbody>
<tr>
<td>ker R</td>
<td>entire R</td>
<td>injective R</td>
</tr>
<tr>
<td>img R</td>
<td>surjective R</td>
<td>simple R</td>
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\[
\text{ker } R = R^\circ \cdot R \\
\text{img } R = R \cdot R^\circ
\]
Functions in one slide

• A function $f$ is a binary relation such that

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<td>“Left” Uniqueness</td>
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</tr>
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<td>$b \in \text{img } f \land b' \in \text{img } f \land f \ a \Rightarrow b = b'$</td>
<td>$\text{id} \subseteq \text{id}$ $(f$ is simple$)$</td>
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<td>Leibniz principle</td>
<td></td>
</tr>
<tr>
<td>$a = a' \Rightarrow f \ a = f \ a'$</td>
<td>$\text{id} \subseteq \text{ker } f$ $(f$ is entire$)$</td>
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• Back to useful “al-djabr” rules (GCs):

$$f \cdot R \subseteq S \equiv R \subseteq f^\circ \cdot S$$

$$R \cdot f^\circ \subseteq S \equiv R \subseteq S \cdot f$$

• Equality:

$$f \subseteq g \equiv f = g \equiv f \supseteq g$$
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$$R \cdot f^\circ \subseteq S \iff R \subseteq S \cdot f$$

- Equality:

$$f \subseteq g \iff f = g \iff f \supseteq g$$
Simple relations in one slide

- "Al-djabr" rules for simple $M$:

\[
M \cdot R \subseteq T \ \equiv \ \ (\delta M) \cdot R \ \subseteq \ \ M^\circ \cdot T \tag{4}
\]

\[
R \cdot M^\circ \subseteq T \ \equiv \ \ R \cdot \delta M \subseteq T \cdot M \tag{5}
\]

where

\[
\delta R = \ker R \cap id
\]

(=domain of $R$) is the coreflexive part of $\ker R$.

- Equality

\[
M = N \equiv \ M \subseteq N \land \delta N \subseteq \delta M \tag{6}
\]

follows from (4, 5).
Simple relations in one slide

- “Al-djabr” rules for simple $M$:

\[
M \cdot R \subseteq T \equiv (\delta M) \cdot R \subseteq M^\circ \cdot T \quad (4)
\]

\[
R \cdot M^\circ \subseteq T \equiv R \cdot \delta M \subseteq T \cdot M \quad (5)
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- Equality

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\]

follows from (4, 5).
Predicates PF-transformed

- **Binary** predicates:
  
  \[ R = \llbracket b \rrbracket \equiv (y \ R \ x \equiv b(y, x)) \]

- **Unary** predicates become fragments of \( id \) (coreflexives):
  
  \[ R = \llbracket p \rrbracket \equiv (y \ R \ x \equiv (p \ x) \land x = y) \]

  eg.

  \[ \llbracket 1 \leq x \leq 4 \rrbracket = \]

  ![Diagram showing the range of \( 1 \leq x \leq 4 \)]
Boolean algebra of coreflexives

\[ [p \land q] = [p] \cdot [q] \quad (7) \]
\[ [p \lor q] = [p] \cup [q] \quad (8) \]
\[ [\neg p] = id - [p] \quad (9) \]
\[ [false] = \bot \quad (10) \]
\[ [true] = id \quad (11) \]

Note the very useful fact that \textbf{conjunction} of coreflexives is \textbf{composition}.
Motivation  Obligations  Laplace  PF-transform  LPF/PF  Invariants  PF data  VDM maps  Summary  Concerns  Closing

LPF versus PF-transform

Example

PF-calculation of “partial” implication [5]:

\[ \forall i, j \in \mathbb{Z} \cdot i \geq j \Rightarrow subp(i, j) = i - j \]

where

\[ subp : \mathbb{Z} \times \mathbb{Z} \rightarrow \mathbb{Z} \]
\[ subp(i, j) \triangleq \text{if } i = j \text{ then } 0 \text{ else } 1 + subp(i, j + 1) \]
Simplicity “does it all” — I think

First step — calculate its PF-transform:

\[(i, j) \in \delta Subp \Rightarrow (i - j) Subp (i, j)\]

\[\equiv \{ \text{PF-transform rule } (f b) R (g a) \equiv b(f^\circ \cdot R \cdot g)a \} \]

\[\delta Subp \subseteq (\neg) \cdot Subp\]

\[\equiv \{ \text{converses} \} \]

\[\delta Subp \subseteq Subp^\circ \cdot (\neg)\]

\[\equiv \{ \text{“al-djabr” (simple relations)} \} \]

\[Subp \subseteq (\neg)\]

Second step: calculate \( Subp \subseteq (\neg) \), see overleaf
Does $Subp \subseteq (\text{\textemdash})$ hold?

We draw

$$
subp : \mathbb{Z} \times \mathbb{Z} \to \mathbb{Z}
$$

$$
subp(i, j) \triangleq \begin{cases} 
0 & \text{if } i = j \\
1 + subp(i, j + 1) & \text{else}
\end{cases}
$$

in a “divide & conquer” diagram:

\[
\begin{array}{cccc}
\mathbb{Z} \times \mathbb{Z} & \xrightarrow{D} & 1 + \mathbb{Z} \times \mathbb{Z} \\
\downarrow \text{Subp} & & \downarrow \text{id} + \text{Subp} \\
\mathbb{Z} & \xleftarrow{c} & 1 + \mathbb{Z}
\end{array}
\]

Thus

$$
Subp = \mu X . (c \cdot (id + X) \cdot D)
$$

where

$$
\Delta = \lambda \ x . (x, x)
$$

$$
D = [\Delta \cdot !^\circ, id \times (-1)]^\circ
$$

$$
c = [0, (1+)]
$$
Does $Subp \subseteq (\cdot)$ hold?

We draw

$$
subp : \mathbb{Z} \times \mathbb{Z} \to \mathbb{Z}
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subp(i, j) \triangleq \begin{cases} 0 & \text{if } i = j \\ 1 + subp(i, j + 1) & \text{else} \end{cases}
$$

in a “divide & conquer” diagram:

$$
\begin{aligned}
\mathbb{Z} \times \mathbb{Z} &\xrightarrow{D} 1 + \mathbb{Z} \times \mathbb{Z} \\
\text{Subp} &\downarrow \quad \downarrow \text{id+Subp} \\
\mathbb{Z} &\xleftarrow{c} 1 + \mathbb{Z}
\end{aligned}
$$

where

$$
\begin{aligned}
\Delta &= \lambda x.(x, x) \\
D &= [\Delta \cdot !^\circ, id \times (-1)]^\circ \\
c &= [0, (1+)]
\end{aligned}
$$

Thus

$$
Subp = \mu X.(c \cdot (id + X) \cdot D))
$$
Does $Subp \subseteq (-)$ hold?

Our calculation is based on the **fixpoint rule**:

$$\mu g \subseteq X \iff g \cdot X \subseteq X$$  \hspace{1cm} (12)

as follows

$$
\begin{align*}
Subp & \subseteq (-) \\
& \iff \{ \text{fixpoint rule, for } g \cdot X = c \cdot (id + X) \cdot D \} \\
& c \cdot (id + (-)) \cdot D \subseteq (-) \\
& \equiv \{ \text{unfold } c \text{ and } D \} \\
& [0, (1+) \cdot (-)] \cdot [\Delta \cdot !^\circ, id \times (-1)]^\circ \subseteq (-) \\
& \equiv \{ \text{converses and coproducts} \}
\end{align*}
$$
Calculate implication

\[ 0 \cdot \Delta^\circ \cup (1+) \cdot (-) \cdot (id \times (-1))^\circ \subseteq (-) \]

\[ \equiv \{ \text{“al-djabr”s of } \cup \text{ and functions} \} \]

\[ 0 = (-) \cdot \Delta \]

\[ (1+) \cdot (-) = (-) \cdot (id \times (-1)) \]

\[ \equiv \{ \text{go pointwise} \} \]

\[ 0 = i - i \]

\[ 1 + (i - j) = i - (j - 1) \]

\[ \equiv \{ \text{arithmetics} \} \]

true

In fact, it can be further shown that the implication is an equivalence — let us see how:
The other side of the equivalence

\[ \forall i, j \in \mathbb{Z} \cdot \text{subp}(i, j) = i - j \Rightarrow i \geq j \]

\[ \equiv \{ \text{PF-transform} \} \]

\[ (\neg) \circ \text{Subp} \cap \text{id} \subseteq \delta \text{Subp} \]

\[ \iff \{ \text{Dedekind}; \text{domain is the coreflexive part of kernel} \} \]

\[ ((\neg) \cap \text{Subp}^\circ) \cdot \text{Subp} \subseteq \text{Subp}^\circ \cdot \text{Subp} \]

\[ \equiv \{ \text{converses}; \text{Subp} \subseteq (\neg), \text{as calculated above} \} \]

\[ \text{Subp}^\circ \cdot \text{Subp} \subseteq \text{Subp}^\circ \cdot \text{Subp} \]

\[ \equiv \{ \text{trivial} \} \]

\[ true \]
Proof obligations (PF-transformed)

Let

\[
\begin{align*}
Inv &= \llbracket inv \rrbracket \quad \text{(a coreflexive)} \\
Pre &= \llbracket pre \rrbracket \quad \text{(a coreflexive)} \\
Post &= \llbracket post \rrbracket
\end{align*}
\]

in

\[
Spec \triangleq Post \cdot Pre
\]

and recall eg.

\[
\begin{align*}
\forall a \cdot pre(a) \Rightarrow \exists b \cdot post(a, b) \\
\forall b, a \cdot pre(a) \land post(a, b) \land inv(a) \Rightarrow inv(b)
\end{align*}
\]
Proof obligations (PF-transformed)

Let

\[
Inv = [\text{inv}] \quad \text{(a coreflexive)}
\]
\[
Pre = [\text{pre}] \quad \text{(a coreflexive)}
\]
\[
Post = [\text{post}]
\]

in

\[
Spec \triangleq Post \cdot Pre
\]

and recall eg.

\[
\forall a \cdot pre(a) \Rightarrow \exists b \cdot post(a, b) \quad (13)
\]
\[
\forall b, a \cdot pre(a) \land post(a, b) \land inv(a) \Rightarrow inv(b) \quad (14)
\]

Then
Proof obligations (PF-transformed)

1. **Satisfiability** — (13) PF-transforms to

   \[ \text{Pre} \subseteq \delta \text{Post} \]  \hspace{1cm} (15)

   equivalent to

   \[ \text{Pre} \subseteq \top \cdot \text{Post} \]

2. **Invariants** — (14) PF-transforms to

   \[ \rho (\text{Spec} \cdot \text{Inv}) \subseteq \text{Inv} \]  \hspace{1cm} (16)

   equivalent to

   \[ \text{Spec} \cdot \text{Inv} \subseteq \text{Inv} \cdot \text{Spec} \]  \hspace{1cm} (17)
Proof obligations (PF-transformed)

1. **Satisfiability** — (13) PF-transforms to

   \[ Pre \subseteq \delta \text{Post} \]  

   equivalent to

   \[ Pre \subseteq \top \cdot \text{Post} \]  

2. **Invariants** — (14) PF-transforms to

   \[ \rho(Spec \cdot Inv) \subseteq Inv \]  

   equivalent to

   \[ Spec \cdot Inv \subseteq Inv \cdot Spec \]
Functions
The special case of (17) where Spec is a function \( f \),

\[ f \cdot \text{Inv} \subseteq \text{Inv} \cdot f \]  \hspace{1cm} (18)

maps back to the pointwise

\[ \forall a \cdot \text{inv}(a) \Rightarrow \text{inv}(f(a)) \]  \hspace{1cm} (19)
Invariants in general

In general, let \( A \xrightarrow{\text{Spec}} B \) be a spec over two datatypes \( A \) and \( B \) each with its invariant, say \( \Phi \) and \( \Psi \), respectively. Then (18) generalizes to

\[
\text{Spec} \cdot \Phi \subseteq \Psi \cdot \text{Spec}
\]  

(20)

We will write

\[
\Phi \xrightarrow{\text{Spec}} \Psi
\]

(21)

to mean \( \text{Spec} \cdot \Phi \subseteq \Psi \cdot \text{Spec} \). Thus,

1. invariants can be regarded as types and
2. invariant preservation can be re-written as a type discipline,
eg.

\[
\Phi \xrightarrow{R} \Psi, \quad \Psi \xrightarrow{S} \Gamma
\]

(22)

(composition),
Invariants in general

In general, let $A \xrightarrow{\text{Spec}} B$ be a spec over two datatypes $A$ and $B$ each with its invariant, say $\Phi$ and $\Psi$, respectively. Then (18) generalizes to

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$$\Phi \xrightarrow{S \cdot R} \Gamma$$

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$$\Phi \xrightarrow{R} \Psi, \quad \Psi \xrightarrow{S} \Gamma$$

(22)

(composition),
**Invariants “are” types**

\[
\begin{align*}
\Phi \xrightarrow{R} \Psi, & \quad \Phi' \subseteq \Phi \\
\therefore \Phi' \xrightarrow{R} \Psi
\end{align*}
\]

\[
\begin{align*}
\Psi' \subseteq \Psi, & \quad \phi \xrightarrow{R} \psi' \\
\therefore \phi \xrightarrow{R} \psi
\end{align*}
\]

(sub-typing), etc

Compare this **invariants-as-types** PF-theory with

**Quoting [4], p.116**

*The valid objects of Datec are those which (...) satisfy inv-Datec. This has a profound consequence for the type mechanism of the notation. (...) The inclusion of a sub-typing mechanism which allows truth-valued functions forces the type checking here to rely on proofs.*
Invariants “are” types

\[
\begin{align*}
\Phi \xrightarrow{R} \Psi, & \quad \Phi' \subseteq \Phi \\
\Phi' \xrightarrow{R} \Psi & \quad (23)
\end{align*}
\]

(sub-typing), etc

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Data structures PF-transformed

- Relational databases resort to the mathematical notion of a relation to model data.
  
  Why not do the same in VDM?

- In the sequel we regard VDM finite mappings \((A \leadsto B)\) as simple relations and resort to “al-djabr” rules to prove invariant preservation.

- Why?
  - No need for induction
  - Proofs don’t even require finiteness
  - (Quite a few) results of the standard VDM theory of mappings
    - extend further to arbitrary binary relations
    - are equivalences, not just implications
Data structures PF-transformed

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VDM mappings are finite **simple** relations

This leads to a PF-transformed mapping theory, eg.

**Mapping comprehension**

\[
\{ g(a) \mapsto f(M(a)) \mid a \in \text{dom } M \}
\]

PF-transforms to

\[
f \cdot M \cdot g^\circ
\]  \hspace{1cm} (24)

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Need to ensure simplicity of the comprehension, see next slide
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Need to ensure simplicity of the comprehension, see next slide
Mapping comprehension — “simple” simplicity argument

\[ f \cdot M \cdot g^\circ \cdot (f \cdot M \cdot g^\circ)^\circ \subseteq id \]
\[ \equiv \{ \text{converses} \} \]
\[ f \cdot M \cdot g^\circ \cdot g \cdot M^\circ \cdot f^\circ \subseteq id \]
\[ \equiv \{ \text{“al-djabr”} \} \]
\[ M \cdot g^\circ \cdot g \cdot M^\circ \subseteq f^\circ \cdot f \]
\[ \equiv \{ \text{definition of kernel of a relation} \} \]
\[ \ker (g \cdot M^\circ) \subseteq \ker f \]
\[ \equiv \{ \text{injectivity preorder } R \leq S \equiv \ker S \subseteq \ker R \} \]
\[ f \leq g \cdot M^\circ \]

That is to say, \( M \) satisfies the \( g \rightarrow f \) functional dependency \([6]\) (always fine wherever \( g \) is injective).
Straight from the VDM-SL on-line manual

<table>
<thead>
<tr>
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PF (formal) semantics:

$$[m_1 \uparrow m_2] = [m_2] \rightarrow [m_2], [m_1]$$

which resorts to the relational version of McCarthy conditional:

$$R \rightarrow S, \quad T \overset{\text{def}}{=} (S \cdot \delta R) \cup (T \cdot \neg \delta R)$$
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\llbracket m_1 \uparrow m_2 \rrbracket = \llbracket m_2 \rrbracket \rightarrow \llbracket m_2 \rrbracket , \llbracket m_1 \rrbracket
\]

which resorts to the relational version of McCarthy conditional:

\[
R \rightarrow S , \ T \overset{\text{def}}{=} (S \cdot \delta R) \cup (T \cdot \neg \delta R)
\]
Mapping override

From PF-definition

\[ M \upharpoonright N \overset{\text{def}}{=} N \rightarrow N , \ M \]  \hspace{1cm} (25)

equivalent to

\[ M \upharpoonright N = N \cup M \cdot (\neg \delta N) \]  \hspace{1cm} (26)

it is easy to show

\[ M \upharpoonright M = M \]  \hspace{1cm} (27)
\[ M \upharpoonright \perp = \perp \upharpoonright M = M \]  \hspace{1cm} (28)

More generally, equivalences

\[ N \subseteq M \equiv M \upharpoonright N = M \]  \hspace{1cm} (29)
\[ \delta M \subseteq \delta N \equiv M \upharpoonright N = N \]  \hspace{1cm} (30)

hold.
Override is associative (Lemma 6.7 in [4] — †-ass)

\[
(R \uparrow S) \uparrow P \\
= \{ (25) \text{ twice } \}
\]

\[
P \rightarrow P, \ (S \rightarrow S, \ R) \\
= \{ (26) \text{ twice } \}
\]

\[
P \cup (S \cup R \cdot (\neg \delta S)) \cdot (\neg \delta P) \\
= \{ \text{ distribution ; de Rorgan } \}
\]

\[
P \cup S \cdot (\neg \delta P) \cup R \cdot (\neg (\delta S \cup \delta P)) \\
= \{ (26) ; \text{ domain of override } \}
\]

\[
(S \uparrow P) \cup R \cdot (\neg \delta (S \uparrow P)) \\
= \{ (26) \}
\]

\[
R \uparrow (S \uparrow P)
\]

Important

- Holds for arbitrary relations
- No need of induction
Override is associative (Lemma 6.7 in [4] — $\dagger$-ass)

\[(R \dagger S) \dagger P\]
\[= \{ (25) \text{ twice } \}\]
\[P \rightarrow P, (S \rightarrow S, R)\]
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\[P \cup S \cdot (\neg \delta P) \cup R \cdot (\neg(\delta S \cup \delta P))\]
\[= \{ (26) ; \text{ domain of override } \}\]
\[(S \dagger P) \cup R \cdot (\neg \delta (S \dagger P))\]
\[= \{ (26) \}\]
\[R \dagger (S \dagger P)\]

Important

- Holds for arbitrary relations
- No need of induction
The ubiquitous finite mapping

Usual “design patterns” in VDM modelling:

- **Classification:** \( A \xrightarrow{\sim} B \) where the type of interest is \( A \) and \( B \) is a classifier

  *Cf. recording (partial) equivalence relations [4]:\
  \( \ker M = R^\circ \cdot R \) for \( M \) simple is always a per (partial equivalence relation).*

- **Quantification:** \( \text{Bag } A \triangle A \xrightarrow{\sim} \mathbb{N} \) (bags, orders, invoices etc)

- **Identification:** \( K \xrightarrow{\sim} A \) where \( A \) is the TOI and \( K \) is a space of keys (eg. name-spaces, database entities, objects, etc)

- **Heaps:** \( K \xrightarrow{\sim} F(A, K) \) where \( K \) is an address space (eg. in modelling memory management)
PF-transformed invariants

Typical *invariant patterns* associated to the *identification* design pattern are

- **Referential integrity:**

\[ M \preceq N \quad \text{or} \quad M^\circ \preceq N \]

where \( \preceq \) denotes the *mapping definition* partial order

\[ M \preceq N = \delta M \subseteq \delta N \quad (31) \]

- **Range-wise property:** because the TOI is in the range, a typical VDM invariant pattern arises, \( \forall \ a \in \text{rng} \ M \cdot \psi(a) \) which PF-transforms to

\[ M \subseteq \psi \cdot M \quad (32) \]
CRUD = identification + persistence

CRUD?

Wikipedia

In computing, **CRUD** is an acronym for Create, Read, Update, and Delete. (...) It is used as a shorthand way to refer to the four basic functions of persistence, which is a major part of nearly all computer software.

**CRUD on mapping** $M$:

- **Create**($N$): $M \mapsto N^\dagger M$
- **Read**($a$): $b$ such that $b \, M \, a$
- **Update**($f, \Phi$): $M \mapsto M^\dagger f \cdot M \cdot \Phi$
- **Delete**($\Phi$): $M \mapsto M \cdot (\neg \Phi)$

Example of proof discharge by PF-calculation: range-wise invariant preservation by (selective) update
CRUD = identification + persistence

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**CRUD on mapping** $M$:

- **Create**($N$): $M \mapsto N \uparrow M$
- **Read**($a$): $b$ such that $b \cdot M = a$
- **Update**($f, \Phi$): $M \mapsto M \uparrow f \cdot M \cdot \Phi$
- **Delete**($\Phi$): $M \mapsto M \cdot (\neg \Phi)$

Example of proof discharge by PF-calculation: **range-wise** invariant preservation by (selective) **update**
Selective update

Notation shorthand

\[ M^f_\Phi \triangleq M \uparrow f \cdot M \cdot \Phi \quad (33) \]

Very easy to show:

\[ M^i_d \Phi = M \quad (34) \]
\[ M^f_\perp = M \quad (35) \]
\[ M^f_{id} = f \cdot M \quad (36) \]

Now, how does selective update \((f_\Phi)\) preserve

\[ inv \ M \triangleq M \subseteq \Psi \cdot M \]
Proof discharge by PF-calculation

We have to find conditions for \((\_\_^f\Phi)\) to bear type

\[
\begin{array}{c}
\text{Inv} \\
\rightarrow \\
\text{Inv}
\end{array}
\]

(37)

Since \((\_\_^f\Phi)\) is a function, the proof discharge is easy (19), for all \(M\):

\[
\begin{align*}
\text{inv}(M) & \Rightarrow \text{inv}(M^f_{\Phi}) \\
\equiv & \quad \{ \text{expand } \text{inv}(M) \} \\
M \subseteq \Psi \cdot M & \Rightarrow M^f_{\Phi} \subseteq \Psi \cdot M^f_{\Phi} \\
\equiv & \quad \{ \text{since } \Psi \cdot M \subseteq M \} \\
M = \Psi \cdot M & \Rightarrow M^f_{\Phi} \subseteq \Psi \cdot M^f_{\Phi}
\end{align*}
\]

So we focus on \(M^f_{\Phi} \subseteq \Psi \cdot M^f_{\Phi}\), assuming \(M = \Psi \cdot M\):
Proof discharge by PF-calculation

\[ M_f^\Phi \subseteq \psi \cdot M_f^\Phi \]

\[ \equiv \{ \text{(33) twice} \} \]

\[ M \uparrow f \cdot M \cdot \Phi \subseteq \psi \cdot (M \uparrow (f \cdot M \cdot \Phi)) \]

\[ \equiv \{ M = \psi \cdot M ; \text{distribution} \} \]

\[ (\psi \cdot M) \uparrow (f \cdot (\psi \cdot M) \cdot \Phi) \subseteq (\psi \cdot M) \uparrow (\psi \cdot f \cdot (M \cdot \Phi)) \]

\[ \leftarrow \{ \text{monotonicity} \} \]

\[ f \cdot \psi \subseteq \psi \cdot f \]

\[ \equiv \{ \text{(21) — of course!} \} \]

\[ \psi \xrightarrow{f} \psi \]
Other variations on mappings

Mapping aliasing
In computing, *aliasing* means multiple names for the same data location.

VDM (pointwise)

\[
\text{alias}(a, b, M) \triangleq M \upharpoonright (\text{if } b \in \text{dom } M \text{ then } \{a \mapsto M(b)\} \text{ else } \{\mapsto\})
\]

PF-transform

\[
\text{alias}(a, b, M) \triangleq M \upharpoonright M \cdot b \cdot a^\circ
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where \(a\) and \(b\) are constant functions.
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Aliasing

Notation shorthand

\( M_{a:=b} \) for \( M \uparrow M \cdot b \cdot a^\circ \) (suggestive of eg. regarding \( M \) as a piece of memory and \( a \) and \( b \) variable names or addresses.)

Sample properties

- Identity:

\[
M_{a:=a} = M
\]  \hfill (38)

- Idempotency:

\[
(M_{a:=b})_{a:=b} = M_{a:=b}
\]  \hfill (39)

both instances of

\[
M_{a:=b} = M \equiv M \cdot b \subseteq M \cdot a
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Equating extends aliasing

Let us move on to the **classification** design pattern, and recall the problem of *Recording equivalence relations* [4]:

**Equate $a$ and $b$**

**VDM:**

$$\text{equate}(a, b, M) \triangleq M \upharpoonright \{x \mapsto M(b)) \mid x \in \text{dom } M \land M(x) = M(a)\}$$

**PF-transform**

$$\text{equate}(a, b, M) \triangleq M \upharpoonright M \cdot b \cdot a^\circ \cdot (\ker M)$$

Thus *equate* is an “evolution” of *aliasing*, equivalent to

$$M \upharpoonright (M \cdot b) \cdot (M \cdot a)^\circ \cdot M$$
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Reasoning about *equate*

**Abstraction function**
Two mappings $M, N$ represent the same PER iff

$$\ker M = \ker N$$

(ker is the abstraction function)

**Properties of *equate***
Writing $M_{a\sim b}$ as abbreviation of $M \uparrow (M \cdot b) \cdot (M \cdot a) \circ \cdot M$:

$$M_{a\sim a} = M \quad (41)$$

$$\ker M_{a\sim b} = \ker M_{b\sim a} \quad (42)$$

and so on.
Reasoning about *equate*

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and so on.
• Learn with the other engineering disciplines
• Rôle of PF-patterns (advantage of “writing less symbols”), eg. easier to spot *al-djabr* rule
• Shift from “implication first” to “calculational” logic
  “Chase” equivalence: *bad use of implication-first logic may lead to “50% loss in theory”*
• PF-transform: need for a cultural “shift”?
Inspiration

• John Backus *Algebra of Programs* (1978) [2]
• Binary relations already in Cliff’s thesis (1981) [3]
• Bird-Meertens-Backhouse approach [1]
Context

- **Coalgebraic** semantics for **components** and objects
- Possibly applicable to VDM(++)
- **Invariants** regarded as coreflexive **bisimulations** in the underlying coalgebra theory
- Finite mappings PF-reasoning relates to on-going work in **database** theory “refactoring” [6]
Current work

- Impact of partial predicates in PF-transform (LPP instead of LPF?)
- Foundations: which approach to undefinedness? LPF [5]? Dijkstra/Scholten’s (and variations thereof)? [7]
- Prospect for tool support:
  - RelView (Kiel)
  - 'G’ALCULATOR project (Minho)
Limitations of \textit{RELVIEW}

\begin{itemize}
\item \textit{RELVIEW} only works on relations with finite domains.
\item Relations between elements have to be explicitly defined.
\item Thus, it is very specific and not usable in the general cases.
\item We need a more generic tool . . .
\end{itemize}
Galculator

- *Galculator* implements relation algebra.
- Relational calculus is done by expression manipulation.
- Manipulation is performed by a strategic typed term-rewriting system implemented using Haskell and GADTs.
- Galois connections are used as rewriting rules allowing the exploitation of proofs by indirect equality.
"Algebra (...) is thing causing admiration"

(...) "Mainly because we see often a great Mathematician unable to resolve a question by Geometrical means, and solve it by Algebra, being that same Algebra taken from Geometry, which is thing causing admiration."

— my (literal, not literary) translation of:

(...) Principalmente que vemos algumas vezes, no poder vn gran Mathematico resoluer vna question por medios Geometricos, y resolverla por Algebra, siendo la misma Algebra sacada de la Geometria, ñ es cosa de admiraciõ.

[ Pedro Nunes (1502-1578) in Libro de Algebra en Arithmetica y Geometria, 1567, fols. 270–270v. ]
Closing

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