Reinvigorating pen-and-paper proofs in VDM: the pointfree approach

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Formal methods

Adopting a **formal** notation standard such as VDM-SL isn't enough:

- abstract models involve **conditions** which lead to
- **proof obligations** that need to be discharged

As in other branches of engineering

$$e = m + c$$

that is,

engineering = *model* first, then *calculate* ...

Calculate? Verify?

We know how to **calculate** since the school desk...

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Tradition on “al-djabr” equational reasoning

Examples of “al-djabr” rules: in arithmetics

$$x - z \leq y \equiv x \leq y + z$$

In logics:

$$(x \wedge \neg z) \Rightarrow y \equiv x \Rightarrow (y \vee z)$$

“Al-djabr” rules are known since the 9c. (They are nowadays known as **Galois connections**.)

Question

Can VDM **proof obligations** be *calculated* along the same tradition?

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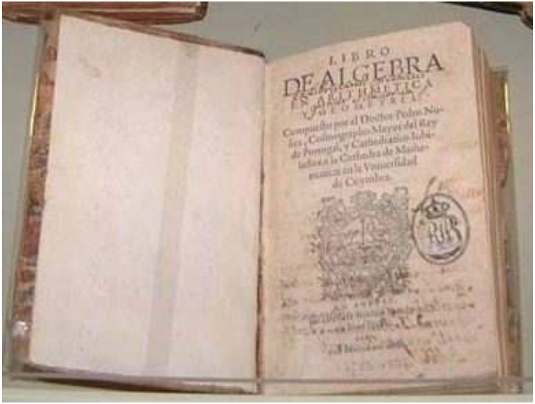
By the way

Nunes' *Libro de Algebra en Arithmetica y Geometria (1567)*



(...) ho inuẽtor desta arte foy hum Mathematico Mouro, cujo nome era Gebre, & ha em alguãs Liuarias hum pequeno tractado Arauigo, que contem os capitulos de q̃ usamos
(fol. a ij r)

Reference to *On the calculus of al-gabr and al-muqâbala* [1] by Abû Abd Allâh Muhamad B. Mûsâ Al-Huwârizmî, a famous 9c Persian mathematician.

[1] Original title: *Kitâb al-muhtasar fi hisab al-gabr wa-almuqâbala*.

Examples of proof obligations

The following are standard in VDM:

- **Satisfiability:** a *pre*/*post* pair is *satisfiable* iff

$$\forall\ a\ \cdot pre(a) \Rightarrow \exists\ b\ \cdot post(a, b) \tag{1}$$

- **Invariants:** in case the *pre*/*post* pair specifies an operation over a state with invariant **inv**,

$$\forall\ a\ \cdot pre(a) \Rightarrow \exists\ b\ \cdot inv(b) \wedge post(a, b) \tag{2}$$

  Moreover, invariants are to be maintained:

$$\forall\ b, a\ \cdot pre(a) \wedge post(a, b) \wedge inv(a) \Rightarrow inv(b) \tag{3}$$

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Impact of (universal) quantification

Quantifiers:

- $\exists$ — easy to discharge (eg. by counter-examples)
- $\forall$ — hard to calculate with (in general), leading to (complex) inductive proofs.

What can we do about this?

- **Mechanical** proof support is one way
- Investigation of **alternative** calculation methods is another

An analogy:

$$\langle \forall\, x \;:\; 0 < x < 10 : \; x^2 \geq x \rangle$$
$$\langle \int x \;:\; 0 < x < 10 : \; x^2 - x \rangle$$

How has traditional **engineering mathematics** tackled the complexity brought about by $\int$'s and $\partial/\partial x$'s?

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The Laplace transform

$(\mathcal{L}\ f)s = \int_0^\infty e^{-st} f(t)dt$

$f(t)$	$\mathcal{L}(f)$
1	$\frac{1}{s}$
t	$\frac{1}{s^2}$
t^n	$\frac{n!}{s^{n+1}}$
e^{at}	$\frac{1}{s-a}$
etc	

Pierre Laplace (1749-1827)

How it works

t-space	*s*-space
Given problem $y'' + 4y' + 3y = 0$ $y(0) = 3$ $y'(0) = 1$	**Subsidiary equation** $s^2 + 4sY + 3Y = 3s + 13$
Solution of given problem $y(t) = -2e^{-3t} + 5e^{-t}$	**Solution of subs. equation** $Y = \frac{-2}{s+3} + \frac{5}{s+1}$

An "s-space analog" for logical quantification

The pointfree ($\mathcal{PF}$) transform

ϕ	$\mathcal{PF}\ \phi$
$\langle \exists\ a\ ::\ b\ R\ a \wedge a\ S\ c \rangle$	$b(R \cdot S)c$
$\langle \forall\ a, b\ ::\ b\ R\ a \Rightarrow b\ S\ a \rangle$	$R \subseteq S$
$\langle \forall\ a\ ::\ a\ R\ a \rangle$	$id \subseteq R$
$\langle \forall\ x\ ::\ x\ R\ b \Rightarrow x\ S\ a \rangle$	$b(R \setminus S)a$
$\langle \forall\ c\ ::\ b\ R\ c \Rightarrow a\ S\ c \rangle$	$a(S\ /\ R)b$
$b\ R\ a \wedge c\ S\ a$	$(b, c)\langle R, S \rangle a$
$b\ R\ a \wedge d\ S\ c$	$(b, d)(R \times S)(a, c)$
$b\ R\ a \wedge b\ S\ a$	$b\ (R \cap S)\ a$
$b\ R\ a \vee b\ S\ a$	$b\ (R \cup S)\ a$
$(f\ \ b)\ R\ (g\ \ a)$	$b(f^{\circ} \cdot R \cdot g)a$
$\textsc{True}$	$b\ \top\ a$
$\textsc{False}$	$b\ \bot\ a$

A transform for logic and set-theory

An old idea

$$\mathcal{PF}(\text{sets, predicates}) \quad = \quad \text{binary relations}$$

Calculus of binary relations

- 1860 - introduced by De Morgan, embryonic
- 1941 - Tarski's school, cf. *A Formalization of Set Theory without Variables*
- 1980's - coreflexive models of sets (Freyd and Scedrov, Eindhoven school)

Unifying approach

Everything is a (binary) relation

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Binary Relations

Arrow notation

Arrow $A \xrightarrow{R} B$ denotes a binary relation to B (target) from A (source).

Identity of composition

id such that $R \cdot id = id \cdot R = R$

Converse

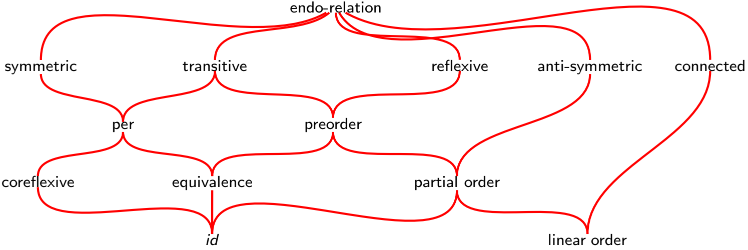
Converse of R — R° such that $a(R^\circ)b$ iff $b\ R\ a$.

Ordering

"$R \subseteq S$ — the "R is at most S" — the obvious $R \subseteq S$ **ordering**.

Binary relation taxonomy

Recall



where a relation $A \xrightarrow{R} A$ is

reflexive:	iff $id_A \subseteq R$
coreflexive:	iff $R \subseteq id_A$
transitive:	iff $R \cdot R \subseteq R$
anti-symmetric:	iff $R \cap R^\circ \subseteq id_A$
symmetric:	iff $R \subseteq R^\circ (\equiv R = R^\circ)$
connected:	iff $R \cup R^\circ = \top$

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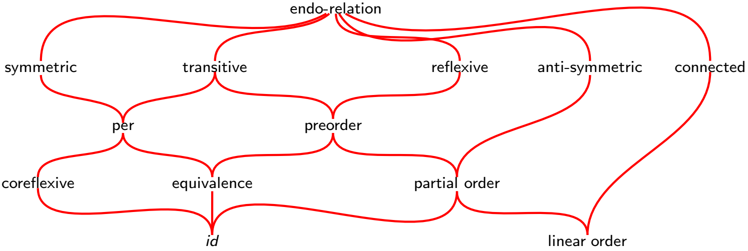

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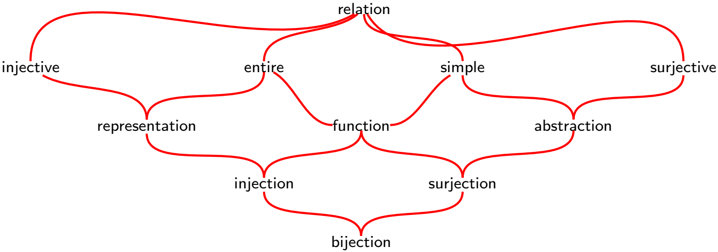
The whole picture:



where

	Reflexive	*Coreflexive*
ker R	**entire** R	**injective** R
img R	**surjective** R	**simple** R

$$\ker R = R^\circ \cdot R$$
$$\mathrm{img}\, R = R \cdot R^\circ$$

Functions in one slide

- A function f is a binary relation such that

Pointwise	Pointfree	
"Left" Uniqueness		
$b\ f\ a \wedge b'\ f\ a \quad \Rightarrow \quad b = b'$	$\mathrm{img}\, f \quad \subseteq \quad id$	(f is simple)
Leibniz principle		
$a = a' \quad \Rightarrow \quad f\ a = f\ a'$	$id \quad \subseteq \quad \ker f$	(f is entire)

- Back to useful "al-djabr" rules (GCs):

$$f \cdot R \subseteq S \quad \equiv R \subseteq f^{\circ} \cdot S$$

$$R \cdot f^{\circ} \subseteq S \quad \equiv R \subseteq S \cdot f$$

- Equality:

$$f \subseteq g \equiv f = g \equiv f \supseteq g$$

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Simple relations in one slide

- "Al-djabr" rules for simple M:

$$M \cdot R \subseteq T \quad \equiv \quad (\delta\, M) \cdot R \quad \subseteq \quad M^\circ \cdot T \tag{4}$$

$$R \cdot M^\circ \subseteq T \quad \equiv \quad R \cdot \delta\, M \quad \subseteq \quad T \cdot M \tag{5}$$

where

$$\delta\, R \quad = \quad \ker R \cap id$$

(=domain of R) is the coreflexive part of $\ker R$.

- **Equality**

$$M = N \equiv \quad M \subseteq N \wedge \delta\, N \subseteq \delta\, M \tag{6}$$

follows from (4, 5).

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Predicates PF-transformed

- **Binary** predicates :

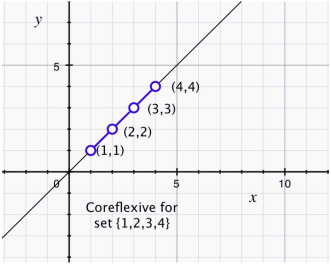
$$R = [\![b]\!] \quad \equiv \quad (y \ R \ x \equiv b(y,x))$$

- **Unary** predicates become fragments of *id* (coreflexives) :

$$R = [\![p]\!] \quad \equiv \quad (y \ R \ x \equiv (p \ x) \land x = y)$$

eg.

$$[\![1 \leq x \leq 4]\!] =$$



Boolean algebra of coreflexives

$$
\begin{aligned}
[\![p \wedge q]\!] &= [\![p]\!] \cdot [\![q]\!] && (7)\\
[\![p \vee q]\!] &= [\![p]\!] \cup [\![q]\!] && (8)\\
[\![\neg p]\!] &= id - [\![p]\!] && (9)\\
[\![false]\!] &= \bot && (10)\\
[\![true]\!] &= id && (11)
\end{aligned}
$$

Note the very useful fact that **conjunction** of coreflexives is **composition**

LPF versus PF-transform

Example

PF-calculation of “partial” implication [5]:

$$\forall\ i, j \in \mathbb{Z}\ \cdot i \geq j \Rightarrow subp(i, j) = i - j$$

where

$$\begin{aligned} &subp : \mathbb{Z} \times \mathbb{Z} \to \mathbb{Z} \\ &subp(i, j) \triangleq \ \ if\ \ i = j\ \ then\ 0\ \ else\ 1 + subp(i, j + 1) \end{aligned}$$

Simplicity "does it all" — I think

First step — calculate its PF-transform:

$$
\begin{array}{ll}
& (i,j) \in \delta\, Subp \Rightarrow (i-j)\ Subp\ (i,j) \\
\equiv & \quad \{\ \text{PF-transform rule}\quad (f\ b)\ R\ (g\ a) \ \equiv\ b(f^\circ \cdot R \cdot g)a \quad \} \\
& \delta\, Subp \ \subseteq\ (-)^\circ \cdot Subp \\
\equiv & \quad \{\ \text{converses}\ \} \\
& \delta\, Subp \ \subseteq\ Subp^\circ \cdot (-) \\
\equiv & \quad \{\ \text{“al-djabr” (simple relations)}\ \} \\
& Subp \ \subseteq\ (-)
\end{array}
$$

Second step: calculate $Subp \subseteq (-)$, see overleaf

Does $Subp \subseteq (-)$ hold?

We draw

$$
\begin{array}{l}
subp : \mathbb{Z} \times \mathbb{Z} \rightarrow \mathbb{Z} \\
subp(i,j) \triangleq \;\; if \;\; i = j \;\; then \;\; 0 \;\; else \;\; 1 + subp(i, j+1)
\end{array}
$$

in a “divide & conqueur” diagram:

$$
\begin{tikzcd}
\mathbb{Z} \times \mathbb{Z} \arrow[r, "D"] \arrow[d, "Subp"'] & 1 + \mathbb{Z} \times \mathbb{Z} \arrow[d, "id+Subp"] \\
\mathbb{Z} & 1 + \mathbb{Z} \arrow[l, "c"]
\end{tikzcd}
$$

where
$$
\begin{array}{lcl}
\Delta & = & \lambda\, x.(x, x) \\
D & = & [\Delta \cdot !^{\circ} \,, id \times (-1)]^{\circ} \\
c & = & [\underline{0} \,, (1+)]
\end{array}
$$

Thus

$$
Subp = \mu X.(c \cdot (id + X) \cdot D))
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Thus

$$Subp \quad = \quad \mu X.(c \cdot (id + X) \cdot D))$$

Does $Subp \subseteq (-)$ hold?

Our calculation is based on the **fixpoint rule**:

$$\mu g \subseteq X \quad \Leftarrow \quad g\ X \subseteq X \tag{12}$$

as follows

$$
\begin{array}{ll}
& Subp \subseteq (-) \\
\Leftarrow & \quad \{ \text{ fixpoint rule , for } g\ X = c \cdot (id + X) \cdot D \ \} \\
& c \cdot (id + (-)) \cdot D \subseteq (-) \\
\equiv & \quad \{ \text{ unfold } c \text{ and } D \ \} \\
& [\underline{0}\ , (1+) \cdot (-)] \cdot [\Delta \cdot !^{\circ}\ , id \times (-1)]^{\circ} \subseteq (-) \\
\equiv & \quad \{ \text{ converses and coproducts } \ \}
\end{array}
$$

Calculate implication

$$
\begin{array}{ll}
& \underline{0} \cdot \Delta^\circ \;\cup\; (1+) \cdot (-) \cdot (id \times (-1))^\circ \quad \subseteq \quad (-) \\
\equiv & \qquad \{ \text{ ``al-djabr''s of } \cup \text{ and functions } \} \\
& \begin{array}{rcl} \underline{0} & = & (-) \cdot \Delta \\ (1+) \cdot (-) & = & (-) \cdot (id \times (-1)) \end{array} \\
\equiv & \qquad \{ \text{ go pointwise } \} \\
& \begin{array}{rcl} 0 & = & i - i \\ 1 + (i - j) & = & i - (j - 1) \end{array} \\
\equiv & \qquad \{ \text{ arithmetics } \} \\
& \mathit{true}
\end{array}
$$

In fact, it can be further shown that the implication is an equivalence — let us see how:

The other side of the equivalence

$$\forall\ i,j \in \mathbb{Z}\ \cdot subp(i,j) = i - j \Rightarrow i \geq j$$

$\equiv$ { PF-transform }

$$(-)^\circ \cdot Subp \cap id \quad \subseteq \quad \delta\, Subp$$

$\Leftarrow$ { Dedekind ; domain is the coreflexive part of kernel }

$$((-)^\circ \cap Subp^\circ) \cdot Subp \quad \subseteq \quad Subp^\circ \cdot Subp$$

$\equiv$ { converses ; $Subp \subseteq (-)$, as calculated above }

$$Subp^\circ \cdot Subp \quad \subseteq \quad Subp^\circ \cdot Subp$$

$\equiv$ { trivial }

$$true$$

Proof obligations (PF-transformed)

Let

$$\begin{array}{rcll} Inv & = & [\![inv]\!] & \text{(a coreflexive)} \\ Pre & = & [\![pre]\!] & \text{(a coreflexive)} \\ Post & = & [\![post]\!] & \end{array}$$

in

$$Spec \triangleq Post \cdot Pre$$

and recall eg.

$$\forall\ a\ \cdot pre(a) \Rightarrow \exists\ b\ \cdot post(a, b) \tag{13}$$

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Proof obligations (PF-transformed)

1. **Satisfiability** — (13) PF-transforms to

$$Pre \quad \subseteq \quad \delta\, Post \tag{15}$$

equivalent to

$$Pre \quad \subseteq \quad \top \cdot Post$$

2. **Invariants** — (14) PF-transforms to

$$\rho\,(Spec \cdot Inv) \subseteq Inv \tag{16}$$

equivalent to

$$Spec \cdot Inv \quad \subseteq \quad Inv \cdot Spec \tag{17}$$

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Proof obligations (PF-transformed)

Functions

The special case of (17) where *Spec* is a function f,

$$f \cdot Inv \quad \subseteq \quad Inv \cdot f \tag{18}$$

maps back to the pointwise

$$\forall\ a \ \cdot inv(a) \Rightarrow inv(f(a)) \tag{19}$$

Invariants in general

In general, let $A \xrightarrow{Spec} B$ be a spec over two datatypes A and B each with its invariant, say Φ and Ψ, respectively. Then (18) generalizes to

$$Spec \cdot \Phi \;\subseteq\; \Psi \cdot Spec \tag{20}$$

We will write

$$\Phi \xrightarrow{Spec} \Psi \tag{21}$$

to mean $Spec \cdot \Phi \;\subseteq\; \Psi \cdot Spec$. Thus,

1. invariants can be regarded as **types** and
2. invariant preservation can be re-written as a **type discipline**, eg.

$$\frac{\Phi \xrightarrow{R} \Psi \quad , \quad \Psi \xrightarrow{S} \Gamma}{\Phi \xrightarrow{S \cdot R} \Gamma} \tag{22}$$

(composition),

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Invariants “are” types

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(sub-typing), etc

Compare this **invariants-as-types** PF-theory with

Quoting [4], p.116

The valid objects of Datec are those which (...) satisfy inv-Datec. This has a profound consequence for the type mechanism of the notation. (...) The inclusion of a sub-typing mechanism which allows truth-valued functions forces the type checking here to rely on proofs.

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Data structures PF-transformed

- Relational databases resort to the mathematical notion of a **relation** to model **data**.

  Why not do the same in VDM?

- In the sequel we regard VDM finite mappings ($A \xrightarrow{\sim} B$) as **simple** relations and resort to "al-djabr" rules to prove invariant preservation

- Why?
  - No need for **induction**
  - Proofs don't even require **finiteness**
  - (Quite a few) results of the standard VDM theory of mappings
    - **extend** further to arbitrary binary relations
    - are **equivalences**, not just implications

Data structures PF-transformed

- Relational databases resort to the mathematical notion of a **relation** to model **data**.

  Why not do the same in VDM?

- In the sequel we regard VDM finite mappings ($A \tilde{\rightarrow} B$) as **simple** relations and resort to "al-djabr" rules to prove invariant preservation

- Why?
  - No need for **induction**
  - Proofs don't even require **finiteness**
  - (Quite a few) results of the standard VDM theory of mappings
    - **extend** further to arbitrary binary relations
    - are **equivalences**, not just implications

VDM mappings are finite **simple** relations

This leads to a PF-transformed mapping theory, eg.

Mapping comprehension

$$\{g(a) \mapsto f(M(a)) \mid a \in dom\ M\}$$

PF-transforms to

$$f \cdot M \cdot g^{\circ} \tag{24}$$

However

Need to ensure simplicity of the comprehension, see next slide

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Mapping comprehension — "simple" simplicity argument

$$
\begin{array}{ll}
& f \cdot M \cdot g^{\circ} \cdot (f \cdot M \cdot g^{\circ})^{\circ} \subseteq id \\
\equiv & \qquad \{ \text{ converses } \} \\
& f \cdot M \cdot g^{\circ} \cdot g \cdot M^{\circ} \cdot f^{\circ} \subseteq id \\
\equiv & \qquad \{ \text{ “al-djabr” } \} \\
& M \cdot g^{\circ} \cdot g \cdot M^{\circ} \subseteq f^{\circ} \cdot f \\
\equiv & \qquad \{ \text{ definition of kernel of a relation } \} \\
& \ker (g \cdot M^{\circ}) \subseteq \ker f \\
\equiv & \qquad \{ \text{ injectivity preorder } R \leq S \equiv \ker S \subseteq \ker R \ \} \\
& f \leq g \cdot M^{\circ}
\end{array}
$$

That is to say, M satisfies the $g \rightarrow f$ *functional dependency* [6] (always fine wherever g is injective).

Straight from the VDM-SL on-line manual

Operator	Name	Semantics description
`m1 † m2`	Override	overrides and merges `m1` with `m2`, i.e. it is like a merge except that `m1` and `m2` need not be compatible; any common elements are as by `m2` (so `m2` overrides `m1`.)

PF (formal) **semantics**:

$$[\![m_1 \dagger m_2]\!] = [\![m_2]\!] \rightarrow [\![m_2]\!] \, , \, [\![m_1]\!]$$

which resorts to the relational version of **McCarthy** conditional:

$$R \rightarrow S \, , \, T \stackrel{\mathrm{def}}{=} (S \cdot \delta R) \cup (T \cdot \neg \delta R)$$

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Mapping override

From PF-definition

$$M \dagger N \stackrel{\text{def}}{=} N \rightarrow N \;,\; M \tag{25}$$

equivalent to

$$M \dagger N = N \cup M \cdot (\neg \delta\, N) \tag{26}$$

it is easy to show

$$M \dagger M = M \tag{27}$$

$$M \dagger \perp = \perp \dagger M = M \tag{28}$$

More generally, **equivalences**

$$N \subseteq M \equiv M \dagger N = M \tag{29}$$

$$\delta\, M \subseteq \delta\, N \equiv M \dagger N = N \tag{30}$$

hold.

Override is associative (Lemma 6.7 in [4] — †-ass)

$$
\begin{array}{cl}
 & (R \dagger S) \dagger P \\
= & \quad \{ \ (25) \text{ twice } \} \\
 & P \to P \ , \ (S \to S \ , \ R) \\
= & \quad \{ \ (26) \text{ twice } \} \\
 & P \cup (S \cup R \cdot (\neg\delta\, S)) \cdot (\neg\delta\, P) \\
= & \quad \{ \ \text{distribution ; de Rorgan } \} \\
 & P \cup S \cdot (\neg\delta\, P) \cup R \cdot (\neg(\delta\, S \cup \delta\, P)) \\
= & \quad \{ \ (26) \text{ ; domain of override } \} \\
 & (S \dagger P) \cup R \cdot (\neg\delta\, (S \dagger P)) \\
= & \quad \{ \ (26) \ \} \\
 & R \dagger (S \dagger P)
\end{array}
$$

Important

- Holds for **arbitrary** relations
- **No** need of **induction**

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Important

- Holds for **arbitrary** relations
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The ubiquitous finite mapping

Usual **"design paterns"** in VDM modelling:

- **Classification:** $A \tilde{\rightharpoonup} B$ where the type of interest is A and B is a classifier

  Cf. recording (partial) equivalence relations [4]: $\ker M = R^\circ \cdot R$ for M simple is always a per (partial equivalence relation).

- **Quantification:** $Bag\ A \triangleq A \tilde{\rightharpoonup} \mathbb{N}$ (bags, orders, invoices etc)
- **Identification:** $K \tilde{\rightharpoonup} A$ where A is the TOI and K is a space of **keys** (eg. name-spaces, database entities, objects, etc)
- **Heaps**: $K \tilde{\rightharpoonup} \mathsf{F}(A, K)$ where K is an address space (eg. in modelling memory management)

PF-transformed invariants

Typical *invariant patterns* associated to the *identification* design pattern are

- **Referential integrity:**

$$M \quad \preceq \quad N \qquad \text{or} \qquad M^\circ \quad \preceq \quad N$$

  where $\preceq$ denotes the **mapping definition** partial order

$$M \preceq N \quad = \quad \delta\, M \subseteq \delta\, N \tag{31}$$

- **Range-wise property:** because the TOI is in the range, a typical VDM invariant pattern arises, $\forall\ a \in rng\ M \ \cdot \psi(a)$ which PF-transforms to

$$M \subseteq \Psi \cdot M \tag{32}$$

CRUD = identification + persistence

CRUD?

Wikipedia

In computing, **CRUD** *is an acronym for Create, Read, Update, and Delete. (...) It is used as a shorthand way to refer to the four basic functions of* **persistence,** *which is a major part of nearly all computer software.*

CRUD on mapping M:

- $Create(N)$: $M \mapsto N \dagger M$
- $Read(a)$: b such that $b\ M\ a$
- $Update(f, \Phi)$: $M \mapsto M \dagger f \cdot M \cdot \Phi$
- $Delete(\Phi)$: $M \mapsto M \cdot (\neg\Phi)$

Example of proof discharge by PF-calculation: **range-wise** invariant preservation by (selective) **update**

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Example of proof discharge by PF-calculation: **range-wise** invariant preservation by (selective) **update**

Selective update

Notation shorthand

$$M^f_\Phi \quad \triangleq \quad M \dagger f \cdot M \cdot \Phi \tag{33}$$

Very easy to show:

$$M^{id}_\Phi = M \tag{34}$$
$$M^f_\perp = M \tag{35}$$
$$M^f_{id} = f \cdot M \tag{36}$$

Now, how does selective update ($\_^f_\Phi$) preserve

$$inv\ M \quad \triangleq \quad M \subseteq \Psi \cdot M$$

Proof discharge by PF-calculation

We have to find conditions for $(\_^f_\Phi)$ to bear type

$$Inv \xrightarrow{(\_^f_\Phi)} Inv \tag{37}$$

Since $(\_^f_\Phi\ )$ is a function, the proof discharge is easy (19), for all M:

$$
\begin{array}{cl}
& inv(M) \quad \Rightarrow \quad inv(M^f_\Phi)) \\
\equiv & \qquad \{ \text{ expand } inv(M) \ \} \\
& M \subseteq \Psi \cdot M \quad \Rightarrow \quad M^f_\Phi \subseteq \Psi \cdot M^f_\Phi \\
\equiv & \qquad \{ \text{ since } \Psi \cdot M \subseteq M \ \} \\
& M = \Psi \cdot M \quad \Rightarrow \quad M^f_\Phi \subseteq \Psi \cdot M^f_\Phi
\end{array}
$$

So we focus on $M^f_\Phi \subseteq \Psi \cdot M^f_\Phi$, assuming $M = \Psi \cdot M$:

Proof discharge by PF-calculation

$$M^f_\Phi \subseteq \Psi \cdot M^f_\Phi$$

$\equiv$ { (33) twice }

$$M \dagger f \cdot M \cdot \Phi \subseteq \Psi \cdot (M \dagger f \cdot M \cdot \Phi)$$

$\equiv$ { $M = \Psi \cdot M$; distribution }

$$(\Psi \cdot M) \dagger f \cdot (\Psi \cdot M) \cdot \Phi \subseteq (\Psi \cdot M) \dagger (\Psi \cdot f \cdot M \cdot \Phi)$$

$\Leftarrow$ { monotonicity }

$$f \cdot \Psi \subseteq \Psi \cdot f$$

$\equiv$ { (21) — of course! }

$$\Psi \xrightarrow{f} \Psi$$

Other variations on mappings

Mapping aliasing

In computing, *aliasing* means multiple names for the same data location.

VDM (pointwise)

$$alias(a, b, M) \triangleq$$
$$M \dagger (\ if\ \ b \in dom\ M\ \ then\ \{a \mapsto M(b)\}\ else\ \{\mapsto\}\ )$$

PF-transform

$$alias(a, b, M) \triangleq\ M \dagger M \cdot \underline{b} \cdot \underline{a}^{\circ}$$

where $\underline{a}$ and $\underline{b}$ are constant functions.

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where $\underline{a}$ and $\underline{b}$ are constant functions.

Aliasing

Notation shorthand

$M_{a:=b}$ for $M \dagger M \cdot \underline{b} \cdot \underline{a}^\circ$ (suggestive of eg. regarding M as a piece of memory and a and b variable names or addresses.)

Sample properties

- **Identity**:

$$M_{a:=a} = M \tag{38}$$

- **Idempotency**:

$$(M_{a:=b})_{a:=b} = M_{a:=b} \tag{39}$$

both instances of

$$M_{a:=b} = M \equiv M \cdot \underline{b} \subseteq M \cdot \underline{a} \tag{40}$$

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Equating extends aliasing

Let us move on to the **classification** design pattern, and recall the problem of *Recording equivalence relations* [4]:

Equate a and b

VDM:

$$
\begin{array}{l}
equate(a, b, M) \triangleq \\
\quad M \dagger \{x \mapsto M(b)) \mid x \in \ dom\ M \wedge M(x) = M(a)\}
\end{array}
$$

PF-transform

$$equate(a, b, M) \triangleq \ M \dagger M \cdot \underline{b} \cdot \underline{a}^\circ \cdot (\ker M)$$

Thus *equate* is an "evolution" of *aliasing*, equivalent to

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Reasoning about *equate*

Abstraction function

Two mappings M, N represent the same PER iff

$$\ker M \quad = \quad \ker N$$

(ker is the abstraction function)

Properties of *equate*

Writing $M_{a \simeq b}$ as abbreviation of $M \dagger (M \cdot \underline{b}) \cdot (M \cdot \underline{a})^{\circ} \cdot M$:

$$M_{a \simeq a} \quad = \quad M \tag{41}$$

$$\ker M_{a \simeq b} \quad = \quad \ker M_{b \simeq a} \tag{42}$$

and so on.

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and so on.

Summary

- Learn with the other engineering disciplines
- Rôle of PF-patterns (advantage of "writing less symbols"), eg. easier to spot *al-djabr* rule
- Shift from "implication first" to "calculational" logic

  "Chase" equivalence : bad use of implication-first logic may lead to "50% loss in theory"

- PF-transform: need for a cultural "shift"?

Inspiration

- John Backus *Algebra of Programs* (1978) [2]
- Binary relations already in Cliff's thesis (1981) [3]
- Bird-Meertens-Backhouse approach [1]

Context

- **Coalgebraic** semantics for **components** and objects
- Possibly applicable to VDM(++)
- **Invariants** regarded as coreflexive **bisimulations** in the underlying coalgebra theory
- Finite mappings PF-reasoning relates to on-going work in **database** theory "refactoring" [6]

Current work

- Impact of partial predicates in PF-transform (LPP instead of LPF?)
- Foundations: which approach to undefinedness? LPF [5]? Dijkstra/Scholten's (and variations thereof)? [7]
- Prospect for tool support:
  - RelView (Kiel)
  - 'G'ALCULATOR project (Minho)

Limitations of *RELVIEW*

- *RELVIEW* only works on relations with finite domains.
- Relations between elements have to be explicitly defined.
- Thus, it is very specific and not usable in the general cases.
- We need a more generic tool ...

Galculator

- *Galculator* implements relation algebra.
- Relational calculus is done by expression manipulation.
- Manipulation is performed by a strategic typed term-rewriting system implemented using **Haskell** and GADTs.
- Galois connections are used as rewriting rules allowing the exploitation of proofs by indirect equality.

Closing

"Algebra (...) is thing causing admiration"

> *(...) "Mainly because we see often a great Mathematician unable to resolve a question by Geometrical means, and solve it by Algebra, being that same Algebra taken from Geometry, which is thing causing admiration."*

— my (literal, not literary) translation of:

> *(...) Principalmente que vemos algumas vezes, no poder vn gran Mathematico resoluer vna question por medios Geometricos, y resolverla por Algebra, siendo la misma Algebra sacada de la Geometria, q̃ es cosa de admiraciõ.*

[**Pedro Nunes** (1502-1578) in **Libro de Algebra en Arithmetica y Geometria**, 1567, fols. 270–270v.]

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