# Domain Universe of VDM-SL

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## Basic Notation

**Definition 1 (Partial Ordering)** A binary relation $\sqsubseteq$ on $D$ is called a *partial ordering* on $D$ iff it is:

1. Reflexive: for all $a \in D, a \sqsubseteq a$.
2. Antisymmetric: for all $a, b \in D, a \sqsubseteq b$ and $b \sqsubseteq a$ imply $a = b$.
3. Transitive: for all $a, b, c \in D, a \sqsubseteq b$ and $b \sqsubseteq c$ imply $a \sqsubseteq c$.

**Definition 2 (Partially ordered set)** A *partially ordered set* $A$ is a pair $(|A|, \sqsubseteq_A)$ where $|A|$ is a set and $\sqsubseteq_A$ is a partial ordering on $|A|$.

## Complete Partial Order Operators

**Definition 3 (Lifting)** For any set $S$, the result of *lifting* $S$ is a cpo $S_\perp$ defined by:

- $|S_\perp| = S \cup \{\perp\}$
- for $a_1, a_2 \in |S_\perp|$, $a_1 \sqsubseteq_{S_\perp} a_2$ iff $a_1 = \perp$ or $a_1 = a_2$.

**Definition 4 (Union-compatible cpo's)** Let $\mathcal{A}$ be a family of cpo's. The family $\mathcal{A}$ is *union-compatible* if:

$$\bigcup \mathcal{A} = (\bigcup\{|A| | A \in \mathcal{A}\}, \bigcup\{\sqsubseteq_A | A \in \mathcal{A}\})$$

is a cpo.

## Finite Subsets and Sequences

**Definition 5 (Finite subsets)** Let $A$ be a flat cpo with $\bot = \bot_A$. Then the cpo of its *finite subsets*, $\mathcal{S}_{CPO}(A)$, is defined as follows:

- $|\mathcal{S}_{CPO}(A)| = \mathbb{F}(|A| \setminus \{\bot\}) \cup \{\bot\}$,
- for $s_1, s_2 \in |\mathcal{S}_{CPO}(A)|$, $s_1 \sqsubseteq_{\mathcal{S}_{CPO}(A)} s_2$ iff $s_1 = \bot$ or $s_1 = s_2$.

**Definition 6 (Finite Sequences)** Let $A$ be a cpo with $\bot = \bot_A$. The cpo of finite sequences of elements of $A$, $\mathcal{L}_{CPO}(A)$, is defined by

- $|\mathcal{L}_{CPO}(A)| = \mathbb{L}(|A| \setminus \{\bot\}) \cup \{\bot\}$
- $\bot \sqsubseteq_{\mathcal{L}_{CPO}(A)} l$ for all $l \in |\mathcal{L}_{CPO}(A)|$, and
  for $l_1, l_2 \in |\mathcal{L}_{CPO}(A)| \setminus \{\bot\}, l_1 \sqsubseteq_{\mathcal{L}_{CPO}(A)} l_2$ iff
  $\underline{\text{len}}(l_1) = \underline{\text{len}}(l_2)$ and
  for $i \in \{1, \ldots, \underline{\text{len}}(l_1)\}, l_1(i) \sqsubseteq_A l_2(i)$.

# Cartesian Product

**Definition 7 (Cartesian product)** Let $A_1, \ldots, A_n$ be cpo's with $\bot = \bot_{A_1} = \ldots = \bot_{A_n}$. Then their *smashed Cartesian product*, $\mathcal{P}_{CPO}(A_1, \ldots, A_n)$, is defined by:

- $|\mathcal{P}_{CPO}(A_1, \ldots, A_n)| = \bigtimes_{i=1}^{n} (|A_i| \setminus \{\bot\}) \cup \{\bot\}$
- $\bot \sqsubseteq_{\mathcal{P}_{CPO}(A_1,\ldots,A_n)} p$ for all $p \in |\mathcal{P}_{CPO}(A_1, \ldots, A_n)|$, and for $(a_1, \ldots, a_n), (a'_1, \ldots, a'_n) \in |\mathcal{P}_{CPO}(A_1, \ldots, A_n)|$, $(a_1, \ldots, a_n) \sqsubseteq_{\mathcal{P}_{CPO}(A_1,\ldots,A_n)} (a'_1, \ldots, a'_n)$ iff $a_1 \sqsubseteq_{A_1} a'_1$ and ... and $a_n \sqsubseteq_{A_n} a'_n$.

## Record space

**Definition 8 (Record space)** Let $id \in Id$ be a VDM-SL identifier, and let $A_1, \ldots, A_n$ be cpo's with $\bot = \bot_{A_1} = \ldots = \bot_{A_n}$. Then the *smashed record cpo*, $\mathcal{R}^{id}_{CPO}(A_1, \ldots, A_n)$, is defined by:

- $|\mathcal{R}^{id}_{CPO}(A_1, \ldots, A_n)| =$
  $(\{id\} \times \bigtimes_{i=1}^{n} (|A_i| \setminus \{\bot\})) \cup \{\bot\}$
- $\bot \sqsubseteq_{\mathcal{R}^{id}_{CPO}(A_1,\ldots,A_n)} r$ for all
  $r \in |\mathcal{R}^{id}_{CPO}(A_1, \ldots, A_n)|$, and
  for $(id, a_1, \ldots, a_n), (id, a_1', \ldots, a_n') \in$
  $|\mathcal{R}^{id}_{CPO}(A_1, \ldots, A_n)|$,
  $(id, a_1, \ldots, a_n) \sqsubseteq_{\mathcal{R}^{id}_{CPO}(A_1,\ldots,A_n)} (id, a_1', \ldots, a_n')$
  iff $a_1 \sqsubseteq_{A_1} a_1'$ and ...and $a_n \sqsubseteq_{A_n} a_n'$.

## Mapping Space

**Definition 9 (Mapping space)** Let $A$ be a flat cpo and $B$ be a cpo with $\bot = \bot_A = \bot_B$. Then the *cpo of smashed mappings* from $A$ to $B$, $\mathcal{M}_{CPO}(A, B)$, is defined as follows:

- $|\mathcal{M}_{CPO}(A, B)| = \mathbb{M}(|A| \setminus \{\bot\}, |B| \setminus \{\bot\}) \cup \{\bot\}$
- $\bot \sqsubseteq_{\mathcal{M}_{CPO}(A,B)} m$ for all $m \in |\mathcal{M}_{CPO}(A, B)|$, and
  for $m_1, m_2 \in |\mathcal{M}_{CPO}(A, B)| \setminus \{\bot\}$,
  $m_1 \sqsubseteq_{\mathcal{M}_{CPO}(A,B)} m_2$ iff
  $\delta_0(m_1) = \delta_0(m_2)$ and for all
  $a \in \delta_0(m_1), m_1(a) \sqsubseteq_B m_2(a)$.

## Function space

**Definition 10 (Function space)** Let $A$ and $B$ be cpo's with $\bot = \bot_A = \bot_B$. Then the *cpo of functions* from $A$ to $B$, $\mathcal{F}_{CPO}(A, B)$, is defined by:

- $|\mathcal{F}_{CPO}(A, B)| = (|A| \to |B|) \cup \{\bot\}$
- $\bot \sqsubseteq_{\mathcal{F}_{CPO}(A,B)} f$ for all $f \in |\mathcal{F}_{CPO}(A, B)|$, and for $f, g \in |\mathcal{F}_{CPO}(A, B)| \setminus \{\bot\}$, $f \sqsubseteq_{\mathcal{F}_{CPO}(A,B)} g$ iff for all $a \in A$, $f(a) \sqsubseteq_B g(a)$.

## Tagging Operator

**Definition 11 (Tagging)** Let $A$ be a cpo with $\bot = \bot_A$. For any $t \in TAG$, *tagging* $A$ with $t$ yields a cpo $\mathcal{T}^t_{CPO}(A)$ defined as follows:

- $|\mathcal{T}^t_{CPO}(A)| = \{(t,a) | a \in (|A| \setminus \{\bot\})\} \cup \{\bot\}$
- $\bot \sqsubseteq_{\mathcal{T}^t_{CPO}(A)} e$ for all $e \in |\mathcal{T}^t_{CPO}(A)|$, and for $(t,a_1),(t,a_2) \in |\mathcal{T}^t_{CPO}(A)|$, $(t,a_1) \sqsubseteq_{\mathcal{T}^t_{CPO}(A)} (t,a_2)$ iff $a_1 \sqsubseteq_A a_2$.

# Basic cpo's

| | | |
|---|---|---|
| Bool-cpo | = | $\mathcal{T}_{CPO}^{\text{'}bool\text{'}}(\mathbb{B}_\perp)$ |
| char-cpo | = | $\mathcal{T}_{CPO}^{\text{'}char\text{'}}(CHAR_\perp)$ |
| nil-cpo | = | $\mathcal{T}_{CPO}^{\text{'}nil\text{'}}(\{\underline{\mathrm{nil}}\}_\perp)$ |
| token-cpo | = | $\mathcal{T}_{CPO}^{\text{'}token\text{'}}(QUOTE_\perp)$ |
| NUM-CPOS | = | $\{\mathcal{T}_{CPO}^{\text{'}num\text{'}}(\mathbb{N}_\perp), \mathcal{T}_{CPO}^{\text{'}num\text{'}}(\mathbb{N}_{1\perp}), \mathcal{T}_{CPO}^{\text{'}num\text{'}}(\mathbb{Z}_\perp),$ |
| | | $\mathcal{T}_{CPO}^{\text{'}num\text{'}}(\mathbb{Q}_\perp), \mathcal{T}_{CPO}^{\text{'}num\text{'}}(\mathbb{R}_\perp)\}$ |
| QUOTE-CPOS | = | $\{\mathcal{T}_{CPO}^{\text{'}quot\text{'}}(\{q\}_\perp) \mid q \in QUOTE\}$ |

$$
\begin{aligned}
U_{\alpha+1} \quad = \quad & U_\alpha \\
\cup \quad & \{\mathcal{U}_{CPO}(\{D_1,\ldots,D_n\}) | D_1,\ldots,D_n \in U_\alpha \wedge \\
& D_1,\ldots,D_n \text{ is a union compatible family}\} \\
\cup \quad & \{\mathcal{T}^{\text{`set'}}_{CPO}(\mathcal{S}_{CPO}(D)) | D \in U_\alpha \wedge D \text{ is flat}\} \\
\cup \quad & \{\mathcal{T}^{\text{`tuple'}}_{CPO}(\mathcal{P}_{CPO}(D_1,\ldots,D_n)) | D_1,\ldots,D_n \in U_\alpha\} \\
\cup \quad & \{\mathcal{T}^{\text{`seq'}}_{CPO}(\mathcal{L}_{CPO}(D)) | D \in U_\alpha\} \\
\cup \quad & \{\mathcal{T}^{\text{`record'}}_{CPO}(\mathcal{R}^{id}_{CPO}(D_1,\ldots,D_n)) \\
& | id \in Id \wedge D_1,\ldots,D_n \in U_\alpha\} \\
\cup \quad & \{\mathcal{T}^{\text{`map'}}_{CPO}(\mathcal{M}_{CPO}(D_1,D_2)) | D_1,D_2 \in U_\alpha \wedge D_1 \text{ is flat}\} \\
\cup \quad & \{\mathcal{T}^{\text{`fun'}}_{CPO}(\mathcal{F}_{CPO}(D_1,D_2)) | D_1,D_2 \in U_\alpha\} \\
\cup \quad & \{\mathcal{T}^{q}_{CPO}(D) | D \in U_\alpha, q \in QUOTE\}.
\end{aligned}
$$

## VDM Domains

$$CPO = \bigcup_{\alpha<\omega_1} U_\alpha.$$

**Construction 12 (Domain universe)** The universe of domains for VDM, $DOM$, is defined by:

$$DOM = \{(((|A|, \sqsubseteq_A), \|A\|) | (|A|, \sqsubseteq_A) \in CPO \wedge \|A\| \subseteq |A| \setminus \{\bot_A\}\}.$$

**Definition 13 (VDM domain operators)** For each of the operators on $CPO$, its extension to a *domain operator* on $DOM$ is defined.

## Further Information

A Naive Domain Universe for VDM (VDM'90)